Some Arithmetic Duality Theorems

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Outline of Part I

Galois cohomology

1. Local duality
   - Duality with respect to a class formation
   - Local duality
   - Euler-Poincaré characteristic
2. An application to Abelian varieties
3. Global duality
   - A duality theorem
   - Poitou-Tate exact sequence
   - Euler-Poincaré characteristic

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Outline of Part II

Etale cohomology

4 Local duality

5 Global cohomology
- Some notations and calculations
- Euler-Poincaré characteristic

6 Artin-Verdier's theorem

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A very brief introduction

Why arithmetic duality??

- In mathematics, solving equations is always interesting.
- e.g. rational points on a variety $V(\mathbb{Q}) = ?$
- Why Galois / étale cohomology?
  - e.g. $H^1_{\acute{e}t}(spec(\mathcal{O}_K), \mathbb{Z}/m\mathbb{Z})^* = Cl(K)/mCl(K)$ for K a number field
  - e.g. $H^1(\mathbb{Q}_p, E)^* = E(\mathbb{Q}_p)$ for $E_{/\mathbb{Q}_p}$ an elliptic curve
- They give some certain *obstructions* of the local-global principal for the problem of rational points.
  - A famous example : $Ш(\mathbb{Q}, E)$ for an elliptic curve.
- Tentative conclusion : the cohomology groups contain important arithmetic information.
- Arithmetic duality theorems may help to understand the question of rational points.
- Allons-y !

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Part I

Galois cohomology

Class formation

Definition

Let G be a profinite group, and C be a G-module (such that $C = \bigcup_{U \leqslant_o G} C^U$). We say that (G, C) is a *class formation* if there exists an isomorphism $inv_U : H^2(U, C) \xrightarrow{\simeq} \mathbb{Q}/\mathbb{Z}$ for each open subgroup $U \leqslant_o G$ with the commutative diagram for $V \leqslant_o U \leqslant_o G$:

$$\begin{array}{ccc} H^2(U, C) & \xrightarrow{Res_{V,U}} & H^2(V, C) \\ {\scriptstyle inv_U}\downarrow\simeq & & {\scriptstyle inv_V}\downarrow\simeq \\ \mathbb{Q}/\mathbb{Z} & \xrightarrow{[U:V]} & \mathbb{Q}/\mathbb{Z} \end{array}$$

and $H^1(U, C) = 0$.

Class formation

- (G, C) = class formation, M = G-module $\rightsquigarrow$ natural pairing:

$$Ext^r_G(M, C) \times H^{2-r}(G, M) \to H^2(G, C) \simeq \mathbb{Q}/\mathbb{Z},$$

$\rightsquigarrow$

$$\alpha^r(G, M) : Ext^r_G(M, C) \to H^{2-r}(G, M)^* = Hom(H^{2-r}(G, M), \mathbb{Q}/\mathbb{Z}),$$

- On the other hand, (G, C) $\rightsquigarrow$ the reciprocity map

$$rec : C^G \to G^{ab}.$$

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Duality with respect to a class formation

Lemma

Let (G, C) be a class formation and M be a finite G-module, then

- *(i)$\alpha^r(G, M)$ is bijective for all $r \geqslant 2$;*
- *(ii)$\alpha^1(G, M)$ is bijective if $\alpha^1(U, \mathbb{Z}/m\mathbb{Z})$ is bijective for all m and all $U \leqslant_o G$;*
- *(iii)$\alpha^0(G, M)$ is surjective (resp. bijective) if $\alpha^0(U, \mathbb{Z}/m\mathbb{Z})$ is surjective (resp. bijective) for all m and all $U \leqslant_o G$.*

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Duality with respect to a class formation

Remark

$P =$ a set of prime numbers
Considering only the P-primary part, a P-class formation will give us a similar lemma.

Notations

- K = non-Archimedean local field
- k = residue field, $char(k) = p$
- $G = Gal(K^s/K)$
- (G, K^{s*}) is a class formation by LCFT

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Local duality

Theorem

Let M be a finite G-module, then

$$\alpha^r(G,M) : Ext^r_G(M, K^{s*}) \to H^{2-r}(G,M)^*$$

is an isomorphism for all r. If $char(K) \nmid \#M$, then $Ext^r_G(M, K^{s*})$ and $H^r(G,M)$ are finite.

Corollary

If $char(K) \nmid \#M$, then there exists a perfect pairing of finite groups (where $M^D = Hom(M, K^{s*})$)

$$H^r(G, M^D) \times H^{2-r}(G,M) \to H^2(G, K^{s*}) \simeq \mathbb{Q}/\mathbb{Z}.$$

Sketch of proof

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- LCFT $\rightsquigarrow$ info. of $rec : K^* \to G^{ab}$,
- $\alpha^1(G, \mathbb{Z}/m\mathbb{Z}) = rec^{(m)} : K^*/K^{*m} \to (G^{ab})^{(m)}$,
- commutative diagram

$$\begin{array}{ccc} & & (G^{ab})_m \\ & \nearrow^{rec_m} & \uparrow \psi \\ \mu_m(K) & \xrightarrow[\alpha^0(G,\mathbb{Z}/m\mathbb{Z})]{} & H^2(G, \mathbb{Z}/m\mathbb{Z})^*, \end{array}$$

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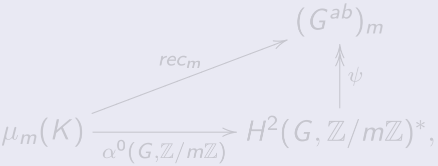

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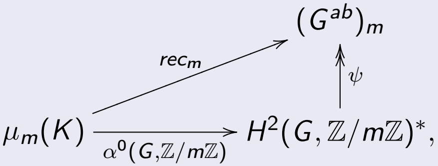

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- In general, ψ: NOT a bijection, BUT in our case $scd(G) = 2 \rightsquigarrow H^3(G, \mathbb{Z}) = 0 \rightsquigarrow \psi$: isomorphism,
- info. of *rec* $\rightsquigarrow$ info. of $\begin{cases} \alpha^0(G, \mathbb{Z}/m\mathbb{Z}) \\ \alpha^1(G, \mathbb{Z}/m\mathbb{Z}) \end{cases}$
- Apply the previous lemma $\Rightarrow$ the statement,
- $\left.\begin{array}{r} \text{spectral sequence} \\ \text{some simple calculations} \end{array}\right\} \rightsquigarrow$ finiteness.
- For the corollary, $char(K) \nmid \#M \rightsquigarrow$ identify $Ext^r_G(M, K^{s*})$ and $H^r(G, M^D)$ by spectral sequence. □

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Euler-Poincaré characteristic

We define the Euler-Poincaré characteristic $\chi(G, M) = \frac{\#H^0(G,M)\cdot\#H^2(G,M)}{\#H^1(G,M)}$, and we have the following formula

Theorem

For M finite of order m such that $char(K) \nmid m$, then

$$\chi(G, M) = |m|_K.$$

Tate's theorem

As an application of the local duality theorem, we get

Theorem (Tate)

Let K be a non-Archimedean local field of characteristic 0, and A be an Abelian variety over K with dual A^t, then there exists a perfect pairing

$$A^t(K) \times H^1(K, A) \to \mathbb{Q}/\mathbb{Z}.$$

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- Finally, Barsotti-Weil formula : $A^t(K) = Ext_K^1(A, \mathbb{G}_m)$ □

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Notations

- K = a global field
- $S \neq \emptyset$ a set of places containing all the archimedean places
- K_S = the maximal extension of K unramified outside S
- $G_S = Gal(K_S/K)$
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A duality theorem

Theorem

Let M be a finite G_S-module, then for any prime number $p \in P$,

$$\alpha^r(G_S, M)(p) : Ext^r_{G_S}(M, C_S)(p) \xrightarrow{\simeq} H^{2-r}(G_S, M)^*(p)$$

is an isomorphism for $r \geqslant 1$. Moreover, if K is a function field then the statement is also true for $r = 0$, in which case P is all the prime numbers.

Proof.

The proof: similar to the local case,
BUT in case $K =$ number field, NOT necessary that $scd(G_S) = 2$,
GCFT $\rightsquigarrow$ info. of $rec \not\Rightarrow$ info. of $\alpha^0(G_S, \mathbb{Z}/p^s\mathbb{Z})$,
that is why the statement is only for $r \geqslant 1$ in this case. □

Notations

- $M^D = Hom(M, K_S^*)$
- $G_v = Gal(K_v^s/K_v) \twoheadrightarrow g_v = Gal(k(v)^s/k(v))$
- $H^r(K_v, M) = \begin{cases} H_T^r(G_v, M), & v \in S_\infty \\ H^r(G_v, M), & v \text{ non-Archimedean} \end{cases}$.
- $H_{un}^r(K_v, M) = im(H^r(g_v, M) \to H^r(G_v, M))$ for $v \notin S_\infty$
- $P_S^r(K, M) = \prod'_{v \in S} H^r(K_v, M)$ restrict prod. wrt. $H_{un}^r(K_v, M)$

Lemma

The image of the homomorphism $H^r(G_S, M) \to \prod_{v \in S} H^r(K_v, M)$ is contained in $P_S^r(K, M)$.

- $\beta_S^r(K, M) : H^r(G_S, M) \to P_S^r(K, M)$
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Theorem (Poitou-Tate)

Let M be a finite G_S-module of order m satisfying $m\mathcal{O}_{K,S} = \mathcal{O}_{K,S}$, then

- *(i)The map $\beta^1_S(K, M)$ is proper, in particular $Ш^1_S(K, M)$ is finite.*
- *(ii)There exists a perfect pairing of finite groups*

$$Ш^1_S(K, M) \times Ш^2_S(K, M^D) \to \mathbb{Q}/\mathbb{Z}.$$

- *(iii)For $r \geqslant 3$, $\beta^r_S(K, M) : H^r(G_S, M) \xrightarrow{\cong} \prod_{v \in S^{\mathbb{R}}} H^r(K_v, M)$ is an isomorphism.*

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Poitou-Tate exact sequence

Theorem (Poitou-Tate)

- *(iv) There is an exact sequence*

$$\begin{array}{ccccccc}
0 \longrightarrow H^0(G_S, M) & \xrightarrow{\beta_S^0} & P_S^0(K, M) & \xrightarrow{\gamma_S^0} & H^2(G_S, M^D)^* & & \\
 & & & & \downarrow & & \\
H^1(G_S, M^D)^* & \xleftarrow{\gamma_S^1} & P_S^1(K, M) & \xleftarrow{\beta_S^1} & H^1(G_S, M) & & \\
\downarrow & & & & & & \\
H^2(G_S, M) & \xrightarrow{\beta_S^2} & P_S^2(K, M) & \xrightarrow{\gamma_S^2} & H^0(G_S, M^D)^* & \longrightarrow & 0.
\end{array}$$

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- (i)Properness of $\beta^1_S(K, M)$: Spectral sequence $\rightsquigarrow$ reduction to simple case,
- Direct calculations for the simple case, finiteness of class group $\Rightarrow$ properness of $\beta^1_S(K, M)$.
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- Symmetry $\Rightarrow$ only need to proof the second half of the sequence,
- $Ext^r_{G_S}(M^D, -)$, $0 \to E_S \to J_S \to C_S \to 0 \rightsquigarrow$ long exact sequence,
- Complicated calculations $\Rightarrow$ $Ext^r_{G_S}(M^D, E_S) = H^r(G_S, M)$ and $Ext^r_{G_S}(M^D, J_S) = P_S^r(K, M)$ for any r,
- Previous duality theorem $\Rightarrow$ $Ext^r_{G_S}(M^D, C_S) = H^r(G_S, M^D)^*$ for $r \geqslant 1$ (the last six terms of the Poitou-Tate sequence). □

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- Complicated calculations $\Rightarrow Ext_{G_S}^r(M^D, E_S) = H^r(G_S, M)$ and $Ext_{G_S}^r(M^D, J_S) = P_S^r(K, M)$ for any r,
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Sketch of proof

Sketch of proof (continued).

- (iii)&(iv): Local duality $\rightsquigarrow$ $\gamma_S^r(K, M^D) : P_S^r(K, M^D) \to H^{2-r}(G_S, M)^*$ is the dual of $\beta_S^{2-r}(K, M) : H^{2-r}(G_S, M) \to P_S^{2-r}(K, M)$,
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Euler-Poincaré characteristic

If $m = \#M$ such that $m\mathcal{O}_{K,S} = \mathcal{O}_{K,S}$, and if S is finite, then $H^r(G_S, M)$ is finite, we define

$$\chi(G_S, M) = \frac{\#H^0(G_S, M) \cdot \#H^2(G_S, M)}{\#H^1(G_S, M)},$$

we have the following formula

Theorem

$$\chi(G_S, M) = \prod_{v \in S_\infty} \frac{\#H^0(G_v, M)}{|m|_v}.$$

Part II

Etale cohomology

Notations

- From now on, all the cohomology groups = étale cohomology groups, "sheaf" = étale sheaf of abelian groups
- R: Henselian DVR, $K = Frac(R)$, $k = R/\mathfrak{m}$ residue field
- $X = spec(R) = \{u, x\}$ where
  - $j : u = spec(K) \to X$ is the generic point
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The local duality theorem

Theorem

Suppose that k is a finite field. Let $\mathcal{F}$ be a constructible sheaf on X, if one of the following conditions holds (1)K is complete, (2)char$(K) = 0$, (3)char$(K) = p$ and $p\mathcal{F} = \mathcal{F}$, then we have a perfect pairing:

$$Ext^r_X(\mathcal{F}, \mathbb{G}_m) \times H^{3-r}_x(X, \mathcal{F}) \to H^3_x(X, \mathbb{G}_m) \simeq \mathbb{Q}/\mathbb{Z}.$$

Corollary

Suppose that k is finite of characteristic p, for a locally constant constructible sheaf $\mathcal{F}$ on X such that $p\mathcal{F} = \mathcal{F}$, then we have a perfect pairing (where $\mathcal{F}^D = \mathcal{H}om_X(\mathcal{F}, \mathbb{G}_m)$)

$$H^r(X, \mathcal{F}^D) \times H^{3-r}_x(X, \mathcal{F}) \to \mathbb{Q}/\mathbb{Z}.$$

Proof of the theorem

Sketch of proof.

1. For sheaves of the form $j_!\mathcal{F}$, we identify the pairing with the local duality of Galois cohomology,
2. For sheaves of the form $i_*\mathcal{F}$, we identify the pairing with the duality of the class formation $(Gal(k^s/k), \mathbb{Z})$,
3. Finally, for general $\mathcal{F}$ we take the cohomology sequence and *Ext* sequence of

$$0 \to j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \to 0$$

and combine the first two cases.
4. For the corollary, $p\mathcal{F} = \mathcal{F} \rightsquigarrow$ identify $Ext^r_X(\mathcal{F}, \mathbb{G}_m)$ and $H^r(X, \mathcal{F}^D)$ by the local-global *Ext* spectral sequence. □

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Notations

- K: a global field
- X
  - $X = spec(\mathcal{O}_K)$ if K is a number field
  - X the unique complete smooth curve with function field K
- Usually, for open subschemes $V \subset U \subseteq X$,
  $j : V \to U =$ the open immersion
  $i : U \setminus V = Z \to U =$ the (reduced) closed immersion
- For a closed point v of X, $\mathcal{O}_v^h =$ Henselization of the stalk of $\mathcal{O}_X$ at v, $K_v = Frac(\mathcal{O}_v^h)$
- For an Archimedean place v, we set $K_v = \mathbb{R}$ or $\mathbb{C}$

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Some calculations

- We can compute $H^r(U, \mathbb{G}_m)$, they are related to the ideal class group (or $Pic(U)$) and the group of unites.
- We can define $H^r_c(U, \mathcal{F}) =$ "cohomology with compact support"
  - in case $K =$ function field, $H^r_c(U, \mathcal{F}) \simeq H^r(X, j_!\mathcal{F})$ is the cohomology with compact support in the classic sense;
  - if $K =$ number field, $H^r_c(U, \mathcal{F})$ is NOT the classic one, but it will give the perfect pairing in the future.
- The important point : $H^r_c(U, \mathcal{F})$ is fixed into a long exact sequence

$$\cdots \to H^r_c(U, \mathcal{F}) \to H^r(U, \mathcal{F}) \to \bigoplus_{v \notin U} H^r(K_v, \mathcal{F}_v) \to \cdots .$$

- Then we can also compute $H^r_c(U, \mathbb{G}_m)$, $H^3_c(U, \mathbb{G}_m) \simeq \mathbb{Q}/\mathbb{Z}$.

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Euler-Poincaré characteristic

For an open subscheme U of X,
$\mathcal{F} \in Sh(U)$ constructible sheaf s.t. $\exists m \in \mathbb{Z}$ satisfying $m\mathcal{F} = 0$ and m invertible on U (i.e. $m\mathcal{O}_v = \mathcal{O}_v$ for all closed point $v \in U$), then $H^r(U, \mathcal{F})$ and $H^r_c(U, \mathcal{F})$ are finite, we define

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Formulae

Theorem

Let $\mathcal{F}$ a constructible sheaf on U such that $m\mathcal{F} = 0$ for a certain integer m invertible on U, then we have the formulae

- *(i)* $\chi(U, \mathcal{F}) = \prod_{v \in S_\infty} \frac{\#\mathcal{F}(K_v)}{\#H^0(K_v, \mathcal{F}) \cdot |\#\mathcal{F}(K^s)|_v}$,
- *(ii)* $\chi_c(U, \mathcal{F}) = \prod_{v \in S_\infty} \#\mathcal{F}(K_v)$.

Sketch of proof.

- First, relate $\chi(U, \mathcal{F})$ with $\chi(V, \mathcal{F}|V)$
- Take a small V s.t. $\mathcal{F}$ is locally constant on V, identify $H^r(V, \mathcal{F})$ with Galois cohomology, and apply the χ global formula for Galois cohomology. □

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Artin-Verdier's theorem

Theorem (Artin-Verdier)

Let $\mathcal{F}$ be a constructible sheaf on U, then we have the following perfect pairing of finite groups

$$Ext_U^r(\mathcal{F}, \mathbb{G}_m) \times H_c^{3-r}(U, \mathcal{F}) \to H_c^3(U, \mathbb{G}_m) \simeq \mathbb{Q}/\mathbb{Z}.$$

Corollary

Let $\mathcal{F}$ be a locally constant constructible sheaf on U such that $m\mathcal{F} = 0$ for a certain integer m invertible on U, then we have the following perfect pairing of finite groups (where $\mathcal{F}^D = \mathcal{H}om_U(\mathcal{F}, \mathbb{G}_m)$)

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Sketch of proof of Artin-Verdier

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- Proof the theorem with assumption $supp(\mathcal{F}) \subseteq Z \subsetneq X$;
- Show that we can replace U by a smaller V, then we can assume $\mathcal{F}$ to be locally constant, killed by m invertible on V;
- Show that we can replace $(U, \mathcal{F})$ by $(U', \mathcal{F}|U')$ with a finite étale covering $U' \to U$, then we can consider only the constant sheaves and assume that K is totally imaginary;

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- With the above assumptions, develop a machine for doing induction on r;
- Show that Ext_U^r and H_c^r vanish if r is large enough or small enough;
- Finally, complete the proof with a supplement argument of Artin-Schreier for the case $char(K) = p$.
- For the corollary, under the assumptions, we identify $Ext_U^r(\mathcal{F}, \mathbb{G}_m)$ and $H^r(U, \mathcal{F}^D)$ by spectral sequence. □

Sketch of proof (continued)

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The End.

- Thank you very much !!
  - Grazie mille !
  - Merci beaucoup !

LIANG, Yong Qi
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